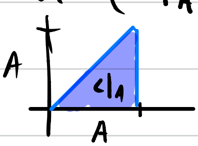
Descriptive Set Theory
Lecture 14

Theorem. Let X be a nonempty perfect Polish space (e.g. $\mathbb{R}$). There is no Baire meas. well-ordering of X, i.e. $\exists$ well-ordering $<$ of X that is BM as a subset of X^2.

Proof. Suppose $<$ is such a well-ordering. An *initial segment* is a subset of X closed downward under $<$.
The idea of the proof is to take the least initial segment that's nonmeager and the K-U will give an even smaller nonmeager initial segment.

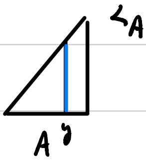
Claim. If $A \subseteq X$ is nonmeager BM initial segment, then $(<|_A) \subseteq X^2$ is nonmeager, where $<|_A := < \cap A^2$.

Proof.



By K-U A^2 is nonmeager, so one of $A^2 = (<|_A) \cup (=|_A) \cup (>|_A)$ must be nonmeager. But $=|_A$ is closed (by Hausdorffness) and has empty interior (because the only rectangles contained in $=|_A$ are of the form $\{x\} \times \{x\}$ and X is perfect), and if $>|_A$ is nonmeager then so is $<|_A$ because the homeomorphism $(x,y) \mapsto (y,x) : X^2 \to X^2$ maps one to the other. $\square$

Since X is nonmeager, we can let A be the least nonmeager initial segment, i.e. $A = X$ or $A = {<^{y_0}} := \{x \in X : x < y_0\}$. By the claim, $<|_A$ is nonmeager. By K-U again, there are nonmeagerly many $y \in A$ s.t. $<^y$ is nonmeager. But $A' := {<^y}$ is a proper initial segment of A, contradicting A being the least nonmeager one. $\square$



Applications of Baire category to top. groups.

Prop. A top. group is Baire $\Leftrightarrow$ it itself is nonmeager.

Proof. HW.

Prop. A subgroup H of a Polish group G is Polish $\Leftrightarrow$ H is closed. In that case, H is nonmeager $\Leftrightarrow$ H is clopen.

Proof. Next HW.

Pettis's Lemma. Let G be a top group and let $A \subseteq G$ be BM.

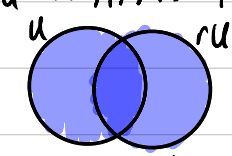
If A is nonmeager, then $A/A := A\cdot A^{-1}$ contains an open neighbourhood of the identity. Same for $A^{-1}A$.

Proof. Since $g \mapsto g^{-1}$ is a homeomorphism, if A is nonmeager, so is A^{-1}, hence it's enough to prove AA^{-1} is nonmeager & $A^{-1}A$ would follow. By the first proposition, A nonmeager $\Rightarrow$ G nonmeager $\Rightarrow$ G is Baire.

Note that $\forall$ subset $B \subseteq G$ and $r \in G$,

$$(*) \qquad r \in B/B \iff r\cdot b_2 = b_1 \text{ for some } b_1, b_2 \in B \iff (r\cdot B) \cap B \neq \emptyset.$$

Let $U =^* A$ open, so $U \neq \emptyset$. Then $U/U = UU^{-1}$ is open hence $U\cdot U^{-1} = \bigcup_{g\in U} Ug^{-1}$. We'll show that $U/U \subseteq A/A$. For any $r \in U/U$, by $(*)$, $rU \cap U \neq \emptyset$



$U \Vdash A$, so $rU \Vdash rA$ by $g \mapsto rg$ is a homeo. Thus, $(rU)\cap U \Vdash (rA)\cap A$, in particular, $(rA)\cap A \neq \emptyset$ since G is Baire. Hence, by $(*)$ again, $r \in A/A$, so $U/U \subseteq A/A$. $\square$

Terry Tao: Weak regularity (=niceness) + group structure $\Rightarrow$ strong regularity.

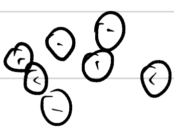
Here is a striking example:

Automatic Continuity. Let G be a Baire top. group and H be a separable top. group. Then any Baire measurable homomorphism $\varphi : G \to H$ is actually continuous.

Proof. Continuity $\Leftrightarrow$ continuity at every pt $g \in G$, i.e. $\forall$ open $V \ni \varphi(g)$ $\exists$ open $U \ni g$ s.t. $\varphi(U) \subseteq V$. Because φ is a homom., this is equivalent to continuity at 1_G; indeed, $\varphi(g)^{-1} \cdot V$ is an open neighbourhood of 1_H, so if $U \ni 1_G$ is open and $\varphi(U) \subseteq \varphi(g)^{-1}(V)$, then $\varphi(gU) = \varphi(g) \cdot \varphi(U) \subseteq V$.

Now fix an open $V \ni 1_H$. We need to find an open $U \ni 1_G$ s.t. $\varphi(U) \subseteq V$. By the continuity of $(x,y) \mapsto xy^{-1}$, $\exists$ open $W \ni 1_H$ s.t. $WW^{-1} \subseteq V$. We'll find $U \ni 1_G$ open s.t. $\varphi(U) \subseteq WW^{-1}$. Suppose for a moment that $\varphi^{-1}(W)$ is nonmeager. Then by Pettis's lemma, $\varphi^{-1}(W) \cdot (\varphi^{-1}(W))^{-1}$ contains an open neighbourhood of 1_G. But because φ is a homom, $\varphi(\varphi^{-1}(W) \cdot (\varphi^{-1}(W))^{-1}) = W \cdot W^{-1}$, so we are done.

But we don't have that $\varphi^{-1}(W)$ is nonmeager. However, if we let $\{h_n\}$ be a ctbl dense set in H, then $H = \bigcup_{n \in \mathbb{N}} W h_n$. Thus, $G = \bigcup_n \varphi^{-1}(W h_n)$, so some $\varphi^{-1}(W h_n)$ has to be nonmeager. The rest of the proof is analogous to the above. $\square$



Measure. Recall that for a top. space X, the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra generated by the open sets.

Def. A Borel measure μ on X is a map $\mu : \mathcal{B}(X) \to [0, \infty]$ s.t. (i) $\mu(\emptyset) = 0$

(ii) μ is countably addive: $\mu\left(\bigsqcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n)$, for any pairwise disjoint $B_0, B_1, B_2, \ldots \in \mathcal{B}(X)$.

It follows that:

(iii) μ is monotone: $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

(iv) μ is countably subaddithe: $\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n\in\mathbb{N}} \mu(B_n)$.

(v) Upward continuous: $A_n \nearrow A$ then $\mu(A_n) \nearrow \mu(A)$.

(vi) Downward continuous: $A_n \searrow A$ and $\mu(A_1) < \infty$, then $\mu(A_n) \searrow \mu(A)$.

Counterexample without $\mu(A_1) < \infty$. Let $A_n := [n, \infty)$, then $A_n \searrow \emptyset$ but $\mu(A_n) = \infty \not\searrow 0$.

We say that μ is finite (probability) (resp. σ-finite) if $\mu(X) < \infty$ ($\mu(X) = 1$) (resp. if $X = \bigcup_n X_n$ s.t. $\mu(X_n) < \infty$). A point $x \in X$ is called an atom of μ if $\mu(x) > 0$, and μ is called atomless or nonatomic if it has no atoms. For a Borel measure μ on X and a Borel function $f: X \to Y$, where Y is another top space and where Borel means that $f^{-1}(\text{Borel})$ is Borel, we define the push-forward measure by f on Y, denoted $f_*\mu$, by $f_*\mu(B) := \mu(f^{-1}(B))$, for each $B \in \mathcal{B}(Y)$.

Examples. (a) For any X and $x \in X$, the Dirac measure at x, denoted δ_x, is the measure that gives 1 to x and 0 to $X \setminus \{x\}$, i.e. $\forall B \subseteq X$,

$$\delta_x(B) = \begin{cases} 1 & \text{if } B \ni x \\ 0 & \text{o.w.} \end{cases}$$

This is a purely atomic measure and is defined on the whole $\mathcal{P}(X)$.

(b) The Lebesgue measure on $\mathbb{R}^d$, $d < \infty$, is denoted by λ_d and is firstly defined on rectangles $I_1 \times I_2 \times \dots \times I_d$, where I_k is an interval, by $\lambda_d(I_1 \times I_2 \times \dots \times I_d) := \ell h(I_1) \cdot \ell h(I_2) \cdot \dots \cdot \ell h(I_d)$, and then extended to $\mathcal{B}(\mathbb{R}^d)$ by Caratheodory's extension theorem, once you verify that λ_d is ctbly additive on the algebra generated by rectangles. λ_d is translation invariant, i.e. $\lambda_d(r+B) = \lambda_d(B) \quad \forall r \in \mathbb{R}^d$.

(c) The unit circle S^1 has a natural rotation invariant probability measure on it by taking the push-forward of the Lebesgue measure on $[0,1)$ by $x \mapsto e^{2\pi i x}$, thinking of S^1 as a subset of $\mathbb{C}$.